Median graphs

CAT(0) cube complexes

(squares), cubes of any dimension

dim 0 : ·
dim 1 : |
dim 2 : □ 1
dim 3 :
etc :

Motivation & history.

- ordered sets $(X, \leq)$
- distributive lattices $(\mathcal{P}(X), \cup, \cap)$

math

non-maths

- social choice theory (voting systems)
- phylogenetics (Petra Schwer: lectures notes on CAT(0) cube complexes).

Started using & studying median graphs

- Birkhoff & Kiss (1947)
- Avann (1961)
- Nebesky (1971)
- Klavžar & Mulder (1999)
- Bandelt & Chepoi (2008)
- Knuth (2008)
- Genevois

Why are median graphs interesting?

- Geometry
- discrete distance and median
- isometries of median (hyperbolic & elliptic)
- group actions (GGT) (Haagerup property)

Graphs:

A graph $X = (V, \sim)$ is a vertex set V along with an adjacency relation $\sim$.

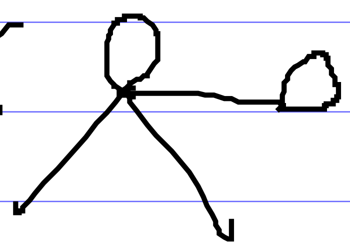
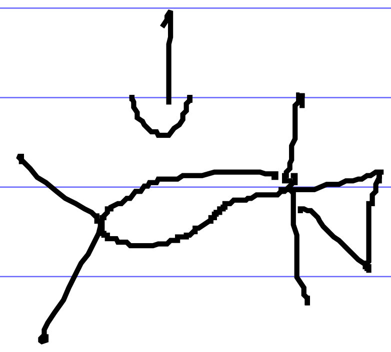
symmetric

→ vertex set: $V(X)$ $\quad$ x, y adjacent $x \sim y$.

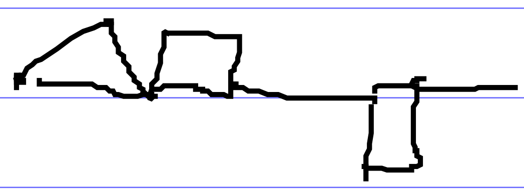
→ edges: $E(X)$. $\quad$ $\{x, y\} \in E(X)$.

X does not contain any loops or multiple edges.

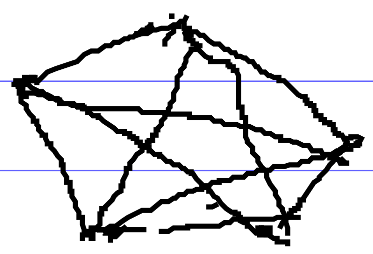
non-eg.:

X  $\quad$ X 

e.g.:



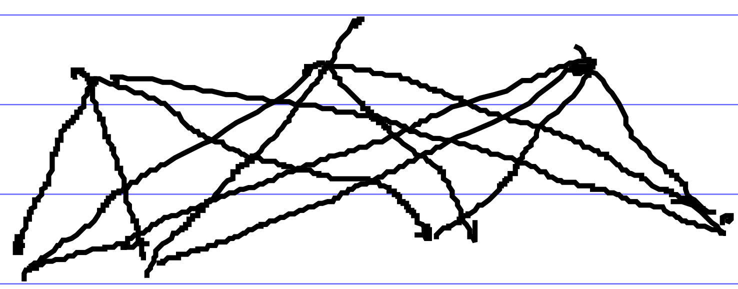
→ complete graphs: K_n $\quad$ $(n \in \mathbb{N}^*)$



K_5

→ complete bipartite graphs: $K_{n,m}$ $\quad$ $(n, m \in \mathbb{N}^*)$

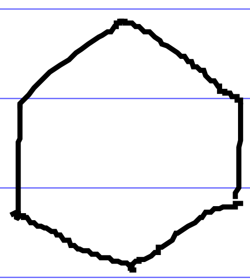
n vertices of Type I



$K_{3,4}$

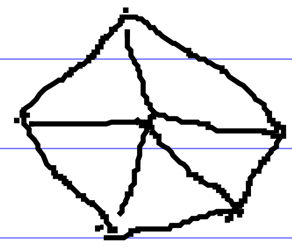
m vertices of type II

→ Cycles: C_n



C_6

→ Wheels : W_n

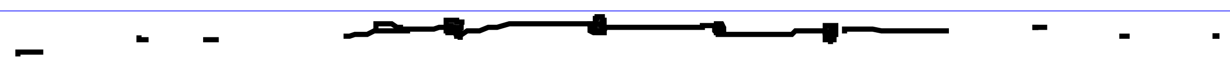


W_5

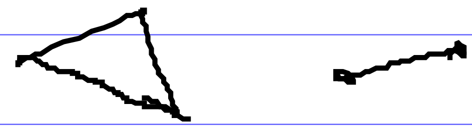
→ paths: $x_0, x_1, \dots, x_n$ $\qquad x_i - x_{i+1} \ \forall i$

→ biinfinite line $\mathbb{E}^1 \to \mathbb{Z}$

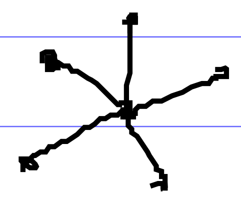
(convention: $X \sim V(X)$)

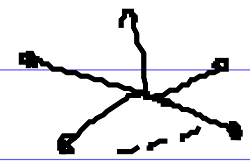


→ connected or non-connected graphs



→ locally finite or non-locally finite





locally infinite.

Distance

connected

Given a graph X, and $x, y \in X$, the distance $d_X(x,y)$ is the length of the shortest path between x and y (geodesic).

$$d_X(x,y) := \min\{\text{length } \gamma, \ \gamma \text{ is a path between } x \text{ and } y \text{ in } X\}.$$

$$d_X(x,y) = 0 \iff x = y \ \checkmark$$

shortest path $\gamma = x_0 = x, x_1, \dots, x_n = y$

between x and y, $\bar{\gamma} = y_0 = y = x_n, \dots, y_n = x_0 = x$

$\Longrightarrow d_X(x,y) = d_X(y,x)$.

(TI): γ_1 geodesic between x and y, and γ_2 geodesic between y and z, the concatenation $\gamma := \gamma_1 \cup \gamma_2$ defines a path between x and z, thus:

$$\begin{aligned} d_X(x,z) &= \text{length}(\gamma') \\ &\leq \text{length}(\gamma) \\ &= \text{length}(\gamma_1) + \text{length}(\gamma_2) \\ &= d_X(x,y) + d_X(y,z). \end{aligned}$$

γ' geodesic between x and z

Subgraphs:

$Y \subseteq X$, (subgraph)

$x, y \in Y$

$x \sim_Y y \iff x \sim_X y$.

Isometrically embedded subgraph

$Y \xhookrightarrow{\text{iso.}} X \iff \forall x,y \in Y,\ d_Y(x,y) = d_X(x,y)$

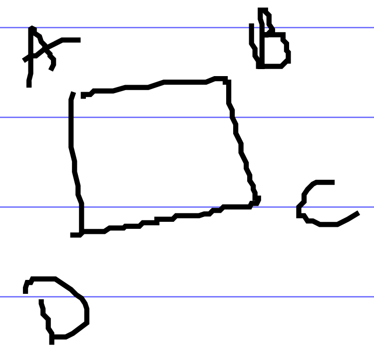
Convex subgraphs

$\forall x,y \in Y$, $\forall \gamma$ geodesic between x and y in X,

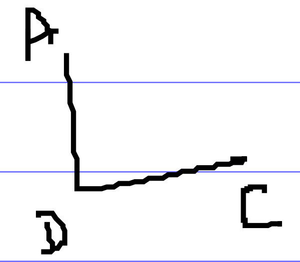
γ lies fully in Y.

~~Rq~~: Convex $\Rightarrow$ isometrically embedded

$\Leftarrow$: X

$X :=$ 

$Y \overset{iso.}{\hookrightarrow} X$

$\underset{X}{\overset{Y}{\cap}} :=$ 

Y is not convex.

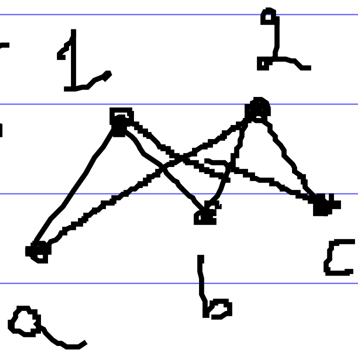
Isomorphism between graphs

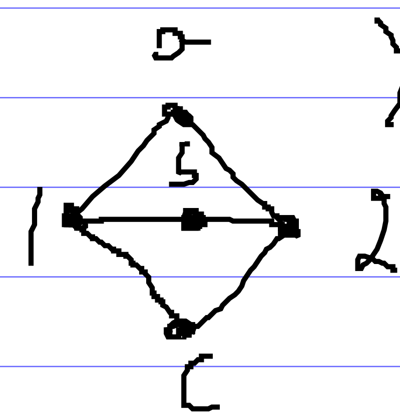
X, Y graphs

$f : X \to Y$ is called a graph isomorphism iff

f sends vertices to vertices
f sends edges to edges
f is bijective
$\forall a, b \in X,\ a \underset{X}{\sim} b \Leftrightarrow f(a) \underset{Y}{\sim} f(b)$

~~e.g.~~:

$K_{2,3} :=$ X 

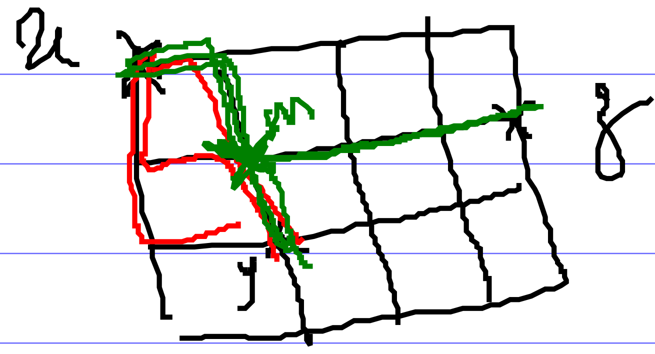
$\cong$  Y

Median graphs

A graph X is said to be median iff $\forall x, y, z \in X$
$\exists! m \in X$ (the median point of the triple x, y, z), s.t.:

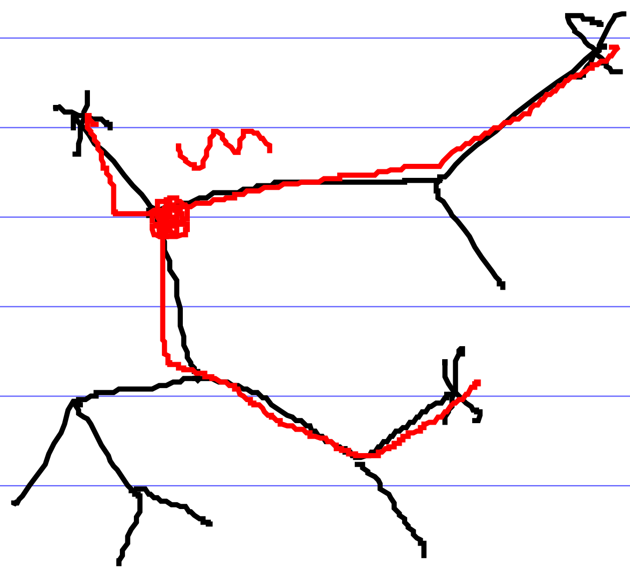
$$\begin{cases} d(x,y) = d(x,m) + d(m,y) \\ d(x,z) = d(x,m) + d(m,z) \\ d(z,y) = d(z,m) + d(m,y) \end{cases}$$

e.g.:



Notation:

$m := \mu(x,y,z)$.



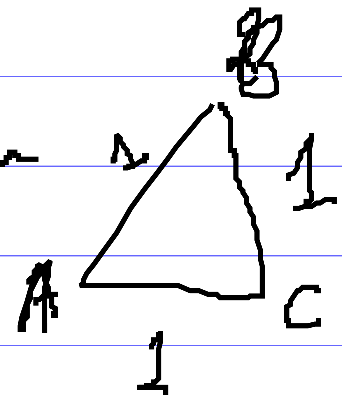
Non-e.g.:

Lemma 1: No median graph contains $K_3 = \triangle$

Lemma 2: No median graph contains $K_{2,3} =$

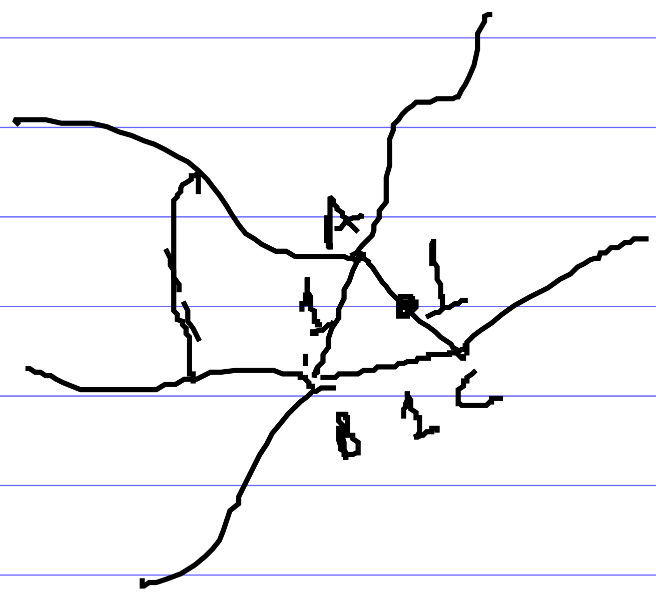
Lemma 3: No median graph contains $C_{2n+1} =$

Lemma 1 pf: K_3 is not median



$\nexists\, \mu_{K_3}(A,B,C)$

$X^{\text{median}} \supseteq K_3$



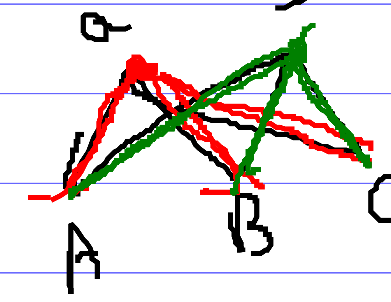
$\mu_X(A,B,C) \nexists$

$= \mu_{K_3}(A,B,C)$

$\square$

Lemma 2 Pf.:

$K_{2,3}$ is not median



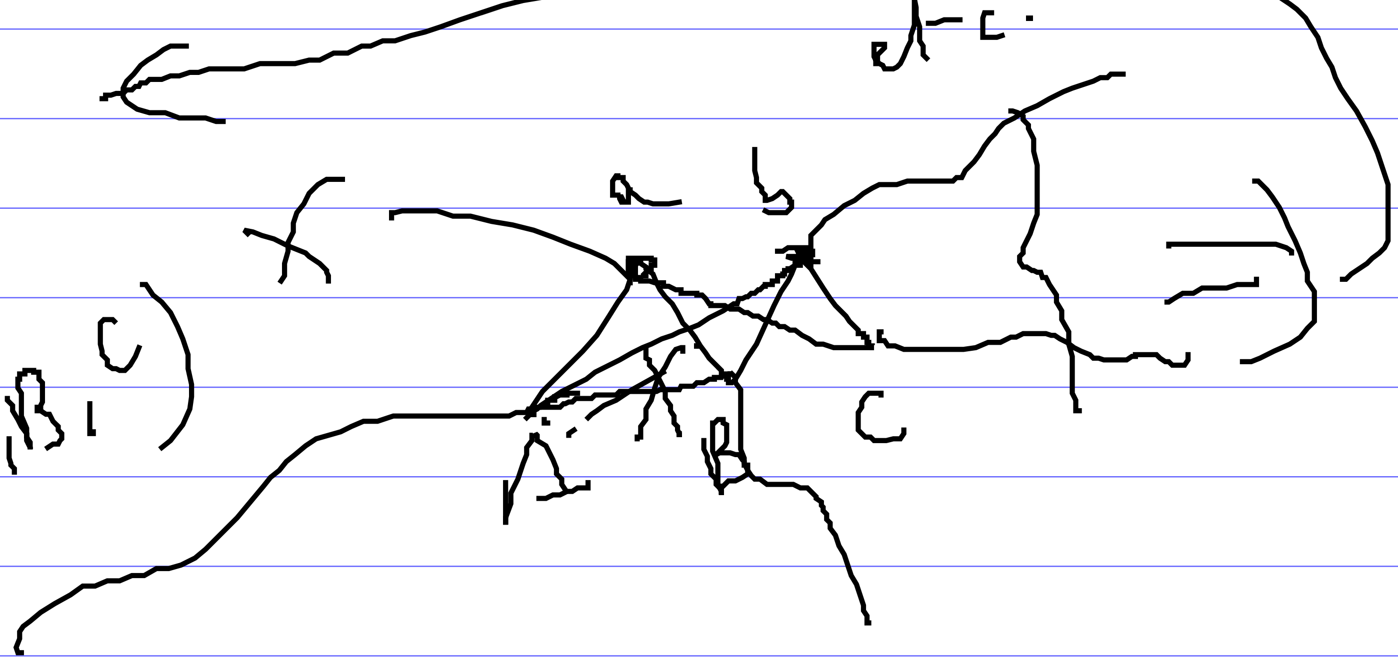
$\mu_{K_{2,3}}(A,B,C) = ?$ $a \neq b$

$\text{geod}(A,B) = \{(A,a,B), (A,b,B)\}$

$\Rightarrow K_{2,3}$ is not median.

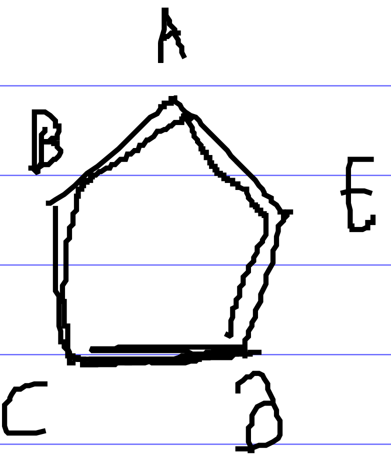
$X \overset{\text{median}}{\cancel{\supseteq}} K_{2,3}$

$\mu_X(A,B,C) = \mu_{K_{2,3}}(A,B,C)$



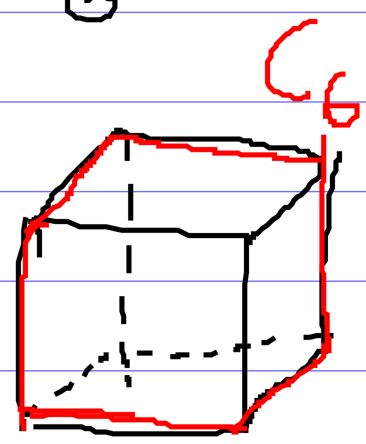
Lemma 3: $C_{2n+1} \not\subseteq X$ median $\forall n \in \mathbb{N}^*$

Pf.: C_{2n+1} not median



$\mu_{C_5}(A,C,D) \not\exists$

(R.q.: $C_6 \subseteq$ median graph.

$Q_3 :=$ 

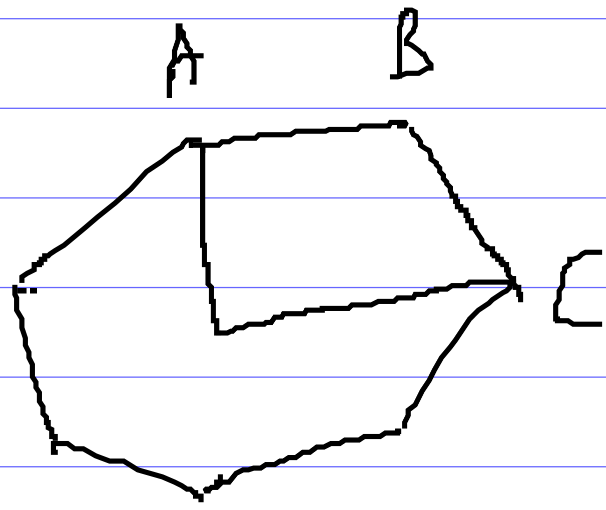
)

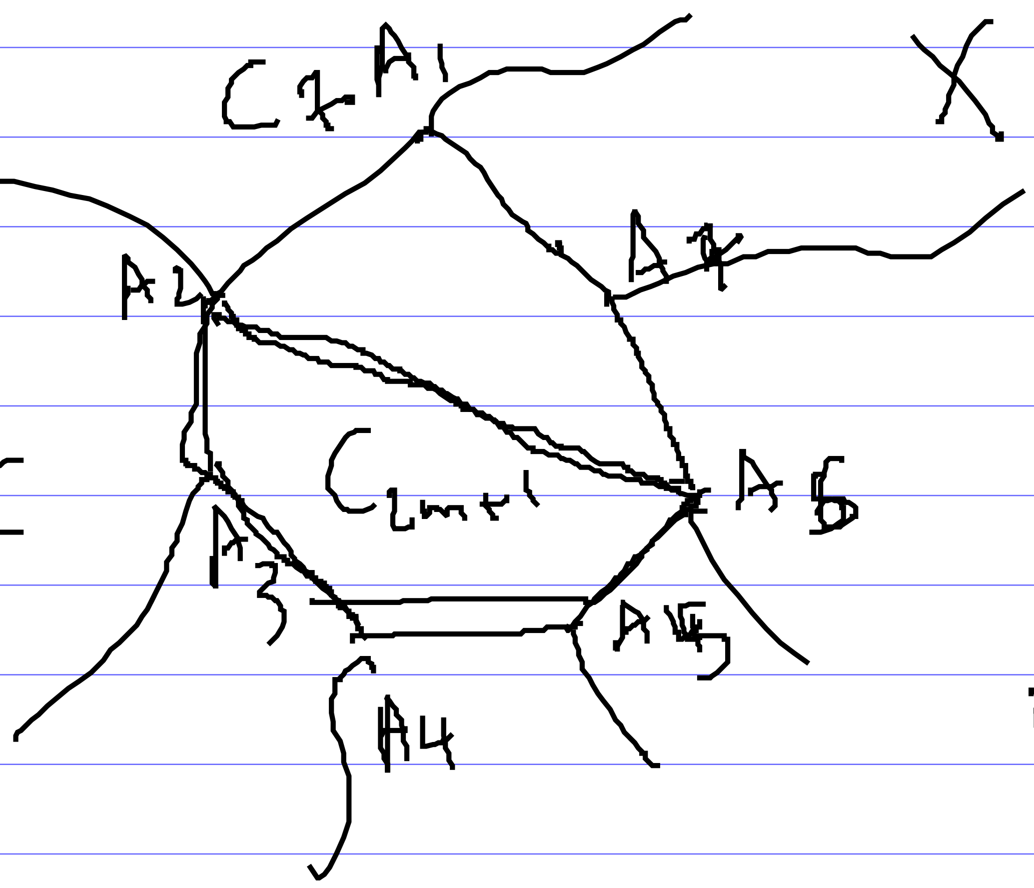
By induction

$\underline{2n+1=3}$ $C_3 = K_3 \notin X^{\text{median}}$

$C_{2n-1} \notin X^{\text{median}}$

Let us show that $C_{2n+1} \notin X^{\text{median}}$





$A_i \sim_X A_j$

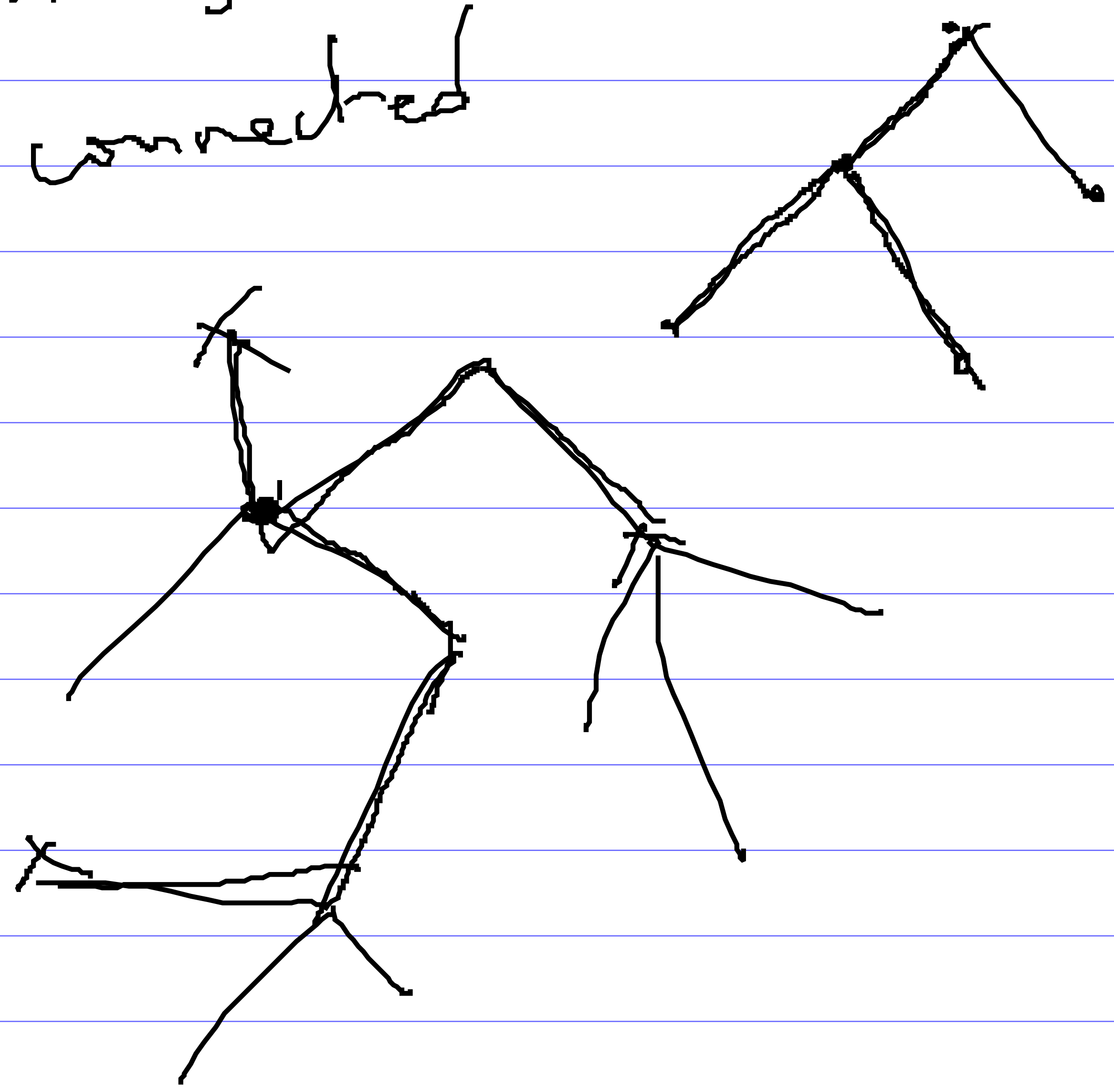
$m<n$

induction hypothesis, this isn't possible.

$\square$

Median graphs:

- trees $\begin{cases} \text{no cycles} \\ \text{connected} \end{cases}$



→ Hypercubes:

S set $\rightsquigarrow Q(S)$

$V(Q(S))$ are in $P(S)$

and $E(Q(S))$: $X, Y \in V(Q(S))$

$$X \underset{Q(S)}{\sim} Y \iff |X \Delta Y| = 1$$

e.g.: $S = \{1, 2, 3\}$

$P(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

$Q(S)$